

AFFINE QUANTUM GROUPS

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Affine quantum groups are certain pseudo-quasitriangular Hopf algebras that arise in mathematical physics in the context of integrable quantum field theory, integrable quantum spin chains, and solvable lattice models. They provide the algebraic framework behind the spectral parameter dependent Yang-Baxter equation

$$(0.1) \quad R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

One can distinguish three classes of affine quantum groups, each leading to a different dependence of the R-matrices on the spectral parameter u : Yangians lead to rational R-matrices, quantum affine algebras lead to trigonometric R-matrices and elliptic quantum groups lead to elliptic R-matrices. We will mostly concentrate on the quantum affine algebras but many results hold similarly for the other classes.

After giving mathematical details about quantum affine algebras and Yangians in the first two sections, we describe how these algebras arise in different areas of mathematical physics in the three following sections. We end with a description of boundary quantum groups which extend the formalism to the boundary Yang-Baxter (reflection) equation.

1. QUANTUM AFFINE ALGEBRAS

1.1. Definition. A quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is a quantization of the enveloping algebra $U(\hat{\mathfrak{g}})$ of an affine Lie algebra (Kac-Moody algebra) $\hat{\mathfrak{g}}$. So we start by introducing affine Lie algebras and their enveloping algebras before proceeding to give their quantizations.

Let \mathfrak{g} be a semisimple finite-dimensional Lie algebra over \mathbb{C} of rank r with Cartan matrix $(a_{ij})_{i,j=1,\dots,r}$, symmetrizable via positive integers d_i , so that $d_i a_{ij}$ is symmetric. In terms of the simple roots α_i , we have

$$a_{ij} = 2 \frac{\alpha_i \cdot \alpha_j}{|\alpha_i|^2} \quad \text{and} \quad d_i = \frac{|\alpha_i|^2}{2}.$$

We can introduce an $\alpha_0 = \sum_{i=1}^r n_i \alpha_i$ in such a way that the extended Cartan matrix $(a_{ij})_{i,j=0,\dots,r}$ is of affine type – that is, it is positive semi-definite of rank r . The integers n_i are referred to as Kac indices. Choosing α_0 to be the highest root of \mathfrak{g} leads to an untwisted affine

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Kac-Moody algebra while choosing α_0 to be the highest short root of \mathfrak{g} leads to a twisted affine Kac-Moody algebra.

One defines the affine Lie algebra $\hat{\mathfrak{g}}$ corresponding to this affine Cartan matrix as the Lie algebra (over \mathbb{C}) with generators H_i, E_i^\pm for $i = 0, 1, \dots, r$ and D and relations

$$(1.1) \quad \begin{aligned} [H_i, E_j^\pm] &= \pm a_{ij} E_i^\pm, \quad [H_i, H_j] = 0, \\ [E_i^+, E_j^-] &= \delta_{ij} H_i, \\ [D, H_i] &= 0, \quad [D, E_i^\pm] = \pm \delta_{i,0} E_i^\pm, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (E_i^\pm)^k E_j^\pm (E_i^\pm)^{1-a_{ij}-k} &= 0, \quad i \neq j. \end{aligned}$$

The E_i^\pm are referred to as Chevalley generators and the last set of relations are known as Serre relations. The generator D is known as the canonical derivation. We will denote the algebra obtained by dropping the generator D by $\hat{\mathfrak{g}}'$.

In applications to physics the affine Lie algebra $\hat{\mathfrak{g}}$ often occurs in an isomorphic form as the loop Lie algebra $\mathfrak{g}[z, z^{-1}] \oplus \mathbb{C} \cdot c$ with Lie product (for untwisted $\hat{\mathfrak{g}}$)

$$(1.2) \quad [Xz^k, Yz^l] = [X, Y]z^{k+l} + \delta_{k,-l}(X, Y)c, \quad \text{for } X, Y \in \mathfrak{g}, \quad k, l \in \mathbb{Z},$$

and c being the central element.

The universal enveloping algebra $U(\hat{\mathfrak{g}})$ of $\hat{\mathfrak{g}}$ is the unital algebra over \mathbb{C} with generators H_i, E_i^\pm for $i = 0, 1, \dots, r$ and D and with relations given by (1.1) where now $[,]$ stands for the commutator instead of the Lie product.

To define the quantization of $U(\hat{\mathfrak{g}})$ one can either define $U_h(\hat{\mathfrak{g}})$ [5] as an algebra over the ring $\mathbb{C}[[h]]$ of formal power series over an indeterminate h or one can define $U_q(\hat{\mathfrak{g}})$ [8] as an algebra over the field $\mathbb{Q}(q)$ of rational functions of q with coefficients in \mathbb{Q} . We will present $U_h(\hat{\mathfrak{g}})$ first.

The *quantum affine algebra* $U_h(\hat{\mathfrak{g}})$ is the unital algebra over $\mathbb{C}[[h]]$ topologically generated by H_i, E_i^\pm for $i = 0, 1, \dots, r$ and D with relations

$$(1.3) \quad \begin{aligned} [H_i, E_j^\pm] &= \pm a_{ij} E_i^\pm, \quad [H_i, H_j] = 0, \\ [E_i^+, E_j^-] &= \delta_{ij} \frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_{-i}}, \\ [D, H_i] &= 0 \quad [D, E_i^\pm] = \pm \delta_{i,0} E_i^\pm, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{matrix} 1-a_{ij} \\ k \end{matrix} \right]_{q_i} (E_i^\pm)^k E_j^\pm (E_i^\pm)^{1-a_{ij}-k} &= 0, \quad i \neq j, \end{aligned}$$

where $q_i = q^{d_i}$ and $q = e^h$. The q -binomial coefficients are defined by

$$(1.4) \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$(1.5) \quad [n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q [1]_q,$$

$$(1.6) \quad \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}.$$

The quantum affine algebra $U_h(\hat{\mathfrak{g}})$ is a Hopf algebra with co-product

$$(1.7) \quad \begin{aligned} \Delta(D) &= D \otimes 1 + 1 \otimes D, \\ \Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\ \Delta(E_i^\pm) &= E_i^\pm \otimes q_i^{-H_i/2} + q_i^{H_i/2} \otimes E_i^\pm, \end{aligned}$$

antipode

$$(1.8) \quad S(D) = -D, \quad S(H_i) = -H_i, \quad S(E_i^\pm) = -q_i^{\mp 1} E_i^\pm,$$

and co-unit

$$(1.9) \quad \epsilon(D) = \epsilon(H_i) = \epsilon(E_i^\pm) = 0.$$

It is easy to see that the classical enveloping algebra $U(\hat{\mathfrak{g}})$ can be obtained from the above by setting $h = 0$, or more formally,

$$U_h(\hat{\mathfrak{g}})/hU_h(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}}).$$

We can also define the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ as the algebra over $\mathbb{Q}(q)$ with generators K_i, E_i^\pm, D for $i = 0, 1, \dots, r$ and relations that are obtained from the ones given above for $U_h(\hat{\mathfrak{g}})$ by setting

$$(1.10) \quad q_i^{H_i/2} = K_i, \quad i = 0, \dots, r.$$

One can also go further to an algebraic formulation over \mathbb{C} in which q is a complex number (with some points including $q = 0$ not allowed). This has the advantage that it becomes possible to specialise for example to q a root of unity, where special phenomena occur.

1.2. Representations. For applications in physics the finite-dimensional representations of $U_h(\hat{\mathfrak{g}}')$ are the most interesting. As will be explained in later sections, these occur for example as particle multiplets in 2-d quantum field theory or as spin Hilbert spaces in quantum spin chains. In the next subsection we will use them to derive matrix solutions to the Yang-Baxter equation.

While for non-affine quantum algebras $U_h(\mathfrak{g})$ the ring of representations is isomorphic to that of the classical enveloping algebra $U(\mathfrak{g})$ (because in fact the algebras are isomorphic, as Drinfeld has pointed

out), the corresponding fact is no longer true for affine quantum groups, except in the case $\hat{\mathfrak{g}} = a_n^{(1)} = \widehat{\mathfrak{sl}_{n+1}}$.

For the classical enveloping algebras $U(\hat{\mathfrak{g}}')$ any finite-dimensional representation of $U(\mathfrak{g})$ also carries a finite-dimensional representation of $U(\hat{\mathfrak{g}}')$. In the quantum case however in general an irreducible representation of $U_h(\hat{\mathfrak{g}}')$ reduces to a sum of representations of $U_h(\mathfrak{g})$.

To classify the finite-dimensional representations of $U_h(\hat{\mathfrak{g}}')$ it is necessary to use a different realization of $U_h(\hat{\mathfrak{g}}')$ that looks more like a quantization of the loop algebra realization (1.2) than the realization in terms of Chevalley generators. In terms of the generators in this alternative realization, which we do not give here because of its complexity, the finite-dimensional representations can be viewed as pseudo-highest weight representations. There is a set of r ‘fundamental’ representations V^a , $a = 1, \dots, r$, each containing the corresponding $U_h(\mathfrak{g})$ fundamental representation as a component, from the tensor products of which all the other finite-dimensional representations may be constructed. The details can be found in [1].

Given some representation $\rho : U_h(\hat{\mathfrak{g}}') \rightarrow \text{End}(V)$ we can introduce a parameter λ with the help of the automorphism τ_λ of $U_h(\hat{\mathfrak{g}}')$ generated by D and given by

$$(1.11) \quad \tau_\lambda(E_i^\pm) = \lambda^{\pm s_i} E_i^\pm, \quad \tau_\lambda(H_i) = H_i, \quad i = 0, \dots, r.$$

Different choices for the s_i correspond to different gradations. Commonly-used are the *homogeneous gradation*, $s_0 = 1, s_1 = \dots = s_r = 0$, and the *principal gradation*, $s_0 = s_1 = \dots = s_r = 1$. We shall also need the *spin gradation* $s_i = d_i^{-1}$. The representations

$$\rho_\lambda = \rho \circ \tau_\lambda$$

play an important role in applications to integrable models where λ is referred to as the (multiplicative) spectral parameter. In applications to particle scattering introduced in a later section it is related to the rapidity of the particle. The generator D can be realized as an infinitesimal scaling operator on λ and thus plays the role of the Lorentz boost generator.

The tensor product representations $\rho_\lambda^a \otimes \rho_\mu^b$ are irreducible generically but become reducible for certain values of λ/μ , a fact which again is important in applications (fusion procedure, particle bound states).

1.3. R-matrices. A Hopf algebra A is said to be *almost cocommutative* if there exists an invertible element $\mathcal{R} \in A \otimes A$ such that

$$(1.12) \quad \mathcal{R}\Delta(x) = (\sigma \circ \Delta(x))\mathcal{R}, \quad \text{for all } x \in A,$$

where $\sigma : x \otimes y \mapsto y \otimes x$ exchanges the two factors in the coproduct. In a *quasitriangular* Hopf algebra this element \mathcal{R} satisfies

$$(1.13) \quad (\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$

and is known as the *universal R-matrix* [See also Article 28]. As a consequence of (1.12) and (1.13) it automatically satisfies the Yang-Baxter equation

$$(1.14) \quad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

For technical reasons, to do with the infinite number of root vectors of $\hat{\mathfrak{g}}$, the quantum affine algebra $U_h(\hat{\mathfrak{g}})$ does not possess a universal R-matrix that is an element of $U_h(\hat{\mathfrak{g}}) \otimes U_h(\hat{\mathfrak{g}})$. However, as pointed out by Drinfeld [5], it possesses a *pseudo-universal* R-matrix $\mathcal{R}(\lambda) \in (U_h(\hat{\mathfrak{g}}') \otimes U_h(\hat{\mathfrak{g}}'))((\lambda))$. The λ is related to the automorphism τ_λ defined in (1.11). When using the homogeneous gradation $\mathcal{R}(\lambda)$ is a formal power series in λ .

When the pseudo-universal R-matrix is evaluated in the tensor product of any two indecomposable finite-dimensional representations ρ_1 and ρ_2 one obtains a numerical R-matrix

$$(1.15) \quad R^{12}(\lambda) = (\rho^1 \otimes \rho^2)\mathcal{R}(\lambda).$$

The entries of these numerical R-matrices are rational functions of the multiplicative spectral parameter λ but when written in terms of the additive spectral parameter $u = \log(\lambda)$ they are trigonometric functions of u and satisfy the Yang-Baxter equation in the form given in (0.1). The matrix

$$\check{R}^{12}(\lambda) = \sigma \circ R^{12}(\lambda)$$

satisfies the intertwining relation

$$(1.16) \quad \check{R}^{12}(\lambda/\mu) \cdot (\rho_\lambda^1 \otimes \rho_\mu^2)\Delta(x) = (\rho_\mu^2 \otimes \rho_\lambda^1)(\Delta(x)) \cdot \check{R}^{12}(\lambda/\mu)$$

for any $x \in U_h(\hat{\mathfrak{g}}')$. It follows from the irreducibility of the tensor product representations that these R-matrices satisfy the Yang-Baxter equations

$$(1.17) \quad \begin{aligned} & (\text{id} \otimes \check{R}^{23}(\mu/\nu))(\check{R}^{13}(\lambda/\nu) \otimes \text{id})(\text{id} \otimes \check{R}^{12}(\lambda/\mu)) \\ &= (\check{R}^{12}(\lambda/\mu) \otimes \text{id})(\text{id} \otimes \check{R}^{13}(\lambda/\nu))(\check{R}^{23}(\mu/\nu) \otimes \text{id}) \end{aligned}$$

or, graphically,

Explicit formulas for the pseudo-universal R-matrices were found by Khoroshkin and Tolstoy. However these are difficult to evaluate explicitly in specific representations so that in practice it is easiest to find the numerical R-matrices $\check{R}^{ab}(\lambda)$ by solving the intertwining relation (1.16). It should be stressed that solving the intertwining relation, which is a linear equation for the R-matrix, is much easier than directly solving the Yang-Baxter equation, a cubic equation.

2. YANGIANS

As remarked by Drinfeld [6], for untwisted $\hat{\mathfrak{g}}$ the quantum affine algebra $U_h(\hat{\mathfrak{g}}')$ degenerates as $h \rightarrow 0$ into another quasi-pseudotriangular Hopf algebra, the *Yangian* $Y(\mathfrak{g})$ [5]. It is associated with R-matrices which are rational functions of the additive spectral parameter u . Its representation ring coincides with that of $U_h(\hat{\mathfrak{g}})$.

Consider a general presentation of a Lie algebra \mathfrak{g} , with generators I_a and structure constants f_{abc} , so that

$$[I_a, I_b] = f_{abc} I_c, \quad \Delta(I_a) = I_a \otimes 1 + 1 \otimes I_a$$

(with summation over repeated indices). The Yangian $Y(\mathfrak{g})$ is the algebra generated by these and a second set of generators J_a satisfying

$$[I_a, J_b] = f_{abc} J_c, \quad \Delta(J_a) = J_a \otimes 1 + 1 \otimes J_a + \frac{1}{2} f_{abc} I_c \otimes I_b.$$

The requirement that Δ be a homomorphism imposes further relations:

$$[J_a, [J_b, I_c]] - [I_a, [J_b, J_c]] = \alpha_{abcdeg} \{I_d, I_e, I_g\}$$

and

$$[[J_a, J_b], [I_l, J_m]] + [[J_l, J_m], [I_a, J_b]] = (\alpha_{abcdeg} f_{lmc} + \alpha_{lmcdg} f_{abc}) \{I_d, I_e, J_g\},$$

where

$$\alpha_{abcdeg} = \frac{1}{24} f_{adi} f_{bej} f_{cgk} f_{ijk}, \quad \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k.$$

When $\mathfrak{g} = \mathfrak{sl}_2$ the first of these is trivial, while for $\mathfrak{g} \neq \mathfrak{sl}_2$, the first implies the second. The co-unit is $\epsilon(I_a) = \epsilon(J_a) = 0$; the antipode is $s(I_a) = -I_a$, $s(J_a) = -J_a + \frac{1}{2}f_{abc}I_cI_b$. The Yangian may be obtained from $U_h(\mathfrak{g}^{(1)})$ by expanding in powers of h . For the precise relationship see [5, 12]. In the spin gradation, the automorphism (1.11) generated by D descends to $Y(\mathfrak{g})$ as $I_a \mapsto I_a$, $J_a \mapsto J_a + uI_a$.

There are two other realizations of $Y(\mathfrak{g})$. The first [13] defines $Y(\mathfrak{gl}_n)$ directly from

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v),$$

where $T_1(u) = T(u) \otimes 1$, $T_2(v) = 1 \otimes T(v)$, and

$$T(u) = \sum_{i,j=1}^n t_{ij}(u) \otimes e_{ij}, \quad t_{ij}(u) = \delta_{ij} + I_{ij}u^{-1} + J_{ij}u^{-2} + \dots,$$

where e_{ij} are the standard matrix units for \mathfrak{gl}_n . The rational R-matrix for the n -dimensional representation of \mathfrak{gl}_n is

$$R(u-v) = 1 - \frac{P}{u-v}, \quad \text{where } P = \sum_{i,j=1}^n e_{ij} \otimes e_{ji}$$

is the transposition operator. $Y(\mathfrak{gl}_n)$ is then defined to be the algebra generated by I_{ij}, J_{ij} , and must be quotiented by the ‘quantum determinant’ at its centre to define $Y(\mathfrak{sl}_n)$. The coproduct takes a particularly simple form,

$$\Delta(t_{ij}(u)) = \sum_{k=1}^n t_{ik}(u) \otimes t_{kj}(u).$$

The third, Drinfeld’s ‘new’ realization of $Y(\mathfrak{g})$ [7], we do not give explicitly here, but we remark that it was in this presentation that Drinfeld found a correspondence between certain sets of polynomials and finite-dimensional irreducible representations of $Y(\mathfrak{g})$, thus classifying these (although not thereby deducing their dimension or constructing the action of $Y(\mathfrak{g})$). As remarked earlier, the structure is as in sect.1.2: $Y(\mathfrak{g})$ representations are in general \mathfrak{g} -reducible, and there is a set of r fundamental $Y(\mathfrak{g})$ -representations, containing the fundamental \mathfrak{g} -representations as components, from which all other representations can be constructed.

3. ORIGINS IN THE QUANTUM INVERSE SCATTERING METHOD

Quantum affine algebras for general $\hat{\mathfrak{g}}$ first appear in [5, 6, 8, 9], but they have their origin in the ‘Quantum Inverse Scattering Method’ (QISM) of the St Petersburg school, and the essential features of $U_h(\widehat{\mathfrak{sl}_2})$ first appear in [11]. In this section we explain how the quantization of

the Lax-pair description of affine Toda theory led to the discovery of the $U_h(\hat{\mathfrak{g}})$ coproduct, commutation relations, and R-matrix. We use the normalizations of [9], in which the H_i are re-scaled so that the Cartan matrix $a_{ij} = \alpha_i \cdot \alpha_j$ is symmetric.

We begin with the affine Toda field equations

$$\partial^\mu \partial_\mu \phi_i = -\frac{m^2}{\beta} \sum_{j=1}^r (e^{\beta a_{ij} \phi_j} - n_i e^{\beta \alpha_0 \cdot \alpha_j \phi_j}) ,$$

an integrable model in \mathbb{R}^{1+1} of r real scalar fields $\phi_i(x, t)$ with a mass parameter m and coupling constant β . Equivalently, we may write $[\partial_x + L_x, \partial_t + L_t] = 0$ for the Lax pair

$$\begin{aligned} L_x(x, t) &= \frac{\beta}{2} \sum_{i=1}^r H_i \partial_t \phi_i + \frac{m}{2} \sum_{i,j=1}^r e^{\frac{\beta}{2} a_{ij} \phi_j} (E_i^+ + E_i^-) + \frac{m}{2} \sum_{j=1}^r e^{\frac{\beta}{2} a_{0j} \phi_j} \left(\lambda E_0^+ + \frac{1}{\lambda} E_0^- \right) \\ L_t(x, t) &= \frac{\beta}{2} \sum_{i=1}^r H_i \partial_x \phi_i + \frac{m}{2} \sum_{i,j=1}^r e^{\frac{\beta}{2} a_{ij} \phi_j} (E_i^+ - E_i^-) + \frac{m}{2} \sum_{j=1}^r e^{\frac{\beta}{2} a_{0j} \phi_j} \left(\lambda E_0^+ - \frac{1}{\lambda} E_0^- \right) \end{aligned}$$

with arbitrary $\lambda \in \mathbb{C}$. The classical integrability of the system is seen in the existence of $r(\lambda, \lambda')$ such that

$$\{T(\lambda) \otimes T(\lambda')\} = [r(\lambda, \lambda'), T(\lambda) \otimes T(\lambda')] ,$$

where $T(\lambda) = T(-\infty, \infty; \lambda)$ and $T(x, y; \lambda) = \mathbf{P} \exp \left(\int_x^y L(\xi; \lambda) d\xi \right)$. Taking the trace of this relation gives an infinity of charges in involution.

Quantization is problematic, owing to divergences in T . The QISM regularizes these by putting the model on a lattice of spacing Δ , defining the lattice Lax operator to be

$$L_n(\lambda) = T((n-1/2)\Delta, (n+1/2)\Delta; \lambda) = \mathbf{P} \exp \left(\int_{(n-\frac{1}{2})\Delta}^{(n+\frac{1}{2})\Delta} L(\xi; \lambda) d\xi \right) .$$

The lattice monodromy matrix is then $T(\lambda) = \lim_{l \rightarrow -\infty, m \rightarrow \infty} T_l^m$ where $T_l^m = L_m L_{m-1} \dots L_{l+1}$, and its trace again yields an infinity of commuting charges, provided that there exists a quantum R-matrix $R(\lambda_1, \lambda_2)$ such that

$$(3.1) \quad R(\lambda_1, \lambda_2) L_n^1(\lambda_1) L_n^2(\lambda_2) = L_n^2(\lambda_2) L_n^1(\lambda_1) R(\lambda_1, \lambda_2),$$

where $L_n^1(\lambda_1) = L_n(\lambda_1) \otimes 1$, $L_n^2(\lambda_2) = 1 \otimes L_n(\lambda_2)$. That R solves the Yang-Baxter equation follows from the equivalence of the two ways of intertwining $L_n(\lambda_1) \otimes L_n(\lambda_2) \otimes L_n(\lambda_3)$ with $L_n(\lambda_3) \otimes L_n(\lambda_2) \otimes L_n(\lambda_1)$.

To compute $L_n(\lambda)$, one uses the canonical, equal-time commutation relations for the ϕ_i and $\dot{\phi}_i$. In terms of the lattice fields

$$p_{i,n} = \int_{(n-\frac{1}{2})\Delta}^{(n+\frac{1}{2})\Delta} \dot{\phi}_i(x) dx, \quad q_{i,n} = \int_{(n-\frac{1}{2})\Delta}^{(n+\frac{1}{2})\Delta} \sum_j e^{\frac{\beta}{2} a_{ij} \phi_j(x)} dx,$$

the only non-trivial relation is $[p_{i,n}, q_{j,n}] = \frac{i\hbar\beta}{2} \delta_{ij} q_{j,n}$, and one finds

$$\begin{aligned} L_n(\lambda) = & \exp\left(\frac{\beta}{2} \sum_i H_i p_{i,n}\right) + \exp\left(\frac{\beta}{4} \sum_j H_j p_{j,n}\right) \frac{m}{2} \left[\sum_i q_{i,n} (E_i^+ + E_i^-) \right. \\ & \left. + \prod_i q_{i,n}^{-n_i} \left(\lambda E_0^+ + \frac{1}{\lambda} E_0^- \right) \right] \exp\left(\frac{\beta}{4} \sum_j H_j p_{j,n}\right) + O(\Delta^2), \end{aligned}$$

the expression used by the St Petersburg school and by Jimbo. We now make the replacement $E_i^\pm \mapsto q^{-H_i/4} E_i^\pm q^{H_i/4}$, where $q = \exp(i\hbar\beta^2/2)$, and compute the $O(\Delta)$ terms in (3.1), which reduce to

$$\begin{aligned} R(z)(H_i \otimes 1 + 1 \otimes H_i) &= (H_i \otimes 1 + 1 \otimes H_i)R(z) \\ R(z)(E_i^\pm \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^\pm) &= (q^{-H_i/2} \otimes E_i^\pm + E_i^\pm \otimes q^{H_i/2}) R(z) \\ R(z)(z^{\pm 1} E_0^\pm \otimes q^{-H_0/2} + q^{H_0/2} \otimes E_0^\pm) &= (q^{-H_0/2} \otimes E_0^\pm + z^{\pm 1} E_0^\pm \otimes q^{H_0/2}) R(z), \end{aligned}$$

where $z = \lambda_1/\lambda_2$. We recognize in these the $U_h(\hat{\mathfrak{g}})$ coproduct and thus the intertwining relations, in the homogeneous gradation. These equations were solved for R in defining representations of non-exceptional \mathfrak{g} by Jimbo in [9].

For $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, it was Kulish and Reshetikhin [11] who first discovered that the requirement that the coproduct must be an algebra homomorphism forces the replacement of the commutation relations of $U(\widehat{\mathfrak{sl}}_2)$ by those of $U_h(\widehat{\mathfrak{sl}}_2)$; more generally it requires the replacement of $U(\hat{\mathfrak{g}})$ by $U_h(\hat{\mathfrak{g}})$.

4. AFFINE QUANTUM GROUP SYMMETRY AND THE EXACT S-MATRIX

In the last section we saw the origins of $U_h(\hat{\mathfrak{g}})$ in the ‘auxiliary’ algebra introduced in the Lax pair. However, the quantum affine algebras also play a second role, as a symmetry algebra. An imaginary-coupled affine Toda field theory based on the affine algebra $\hat{\mathfrak{g}}^\vee$ possesses the quantum affine algebra $U_h(\hat{\mathfrak{g}})$ as a symmetry algebra, where $\hat{\mathfrak{g}}^\vee$ is the Langland dual to $\hat{\mathfrak{g}}$ (the algebra obtained by replacing roots by coroots).

The solitonic particle states in affine Toda theories form multiplets which transform in the fundamental representations of the quantum

affine algebra. Multi-particle states transform in tensor product representations $V^a \otimes V^b$. The scattering of two solitons of type a and b with relative rapidity θ is described by the S -matrix $S^{ab}(\theta) : V^a \otimes V^b \rightarrow V^b \otimes V^a$, graphically represented in figure 1 a). It then follows from the symmetry that the two-particle scattering matrix (S-matrix) for solitons must be proportional to the intertwiner for these tensor product representations, the R matrix:

$$S^{ab}(\theta) = f^{ab}(\theta) \check{R}^{ab}(\theta),$$

with θ proportional to u , the additive spectral parameter. The scalar prefactor $f^{ab}(\theta)$ is not determined by the symmetry but is fixed by other requirements like unitarity, crossing symmetry, and the bootstrap principle.

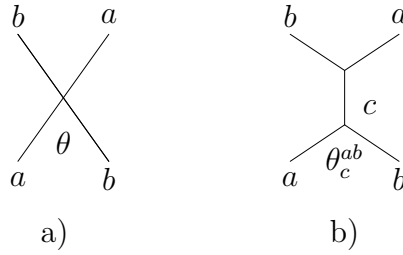


FIGURE 1. a) Graphical representation of a two-particle scattering process described by the S -matrix $S_{ab}(\theta)$. b) At special values θ_{ab}^c of the relative spectral parameter the two particles of types a and b form a bound state of type c .

It turns out that the axiomatic properties of the R -matrices are in perfect agreement with the axiomatic properties of the analytic S -matrix. For example, crossing symmetry of the S -matrix, graphically represented by

(4.1)

is a consequence of the property of the universal R-matrix with respect to the action of the antipode S ,

$$(S \otimes 1)\mathcal{R} = \mathcal{R}^{-1}.$$

An S -matrix will have poles at certain imaginary rapidities θ_c^{ab} corresponding to the formation of virtual bound states. This is graphically represented in figure 1 b). The location of the pole is determined by the masses of the three particles involved,

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos(i\theta_c^{ab}).$$

At the bound state pole the S -matrix will project onto the multiplet V^c . Thus the \check{R} matrix has to have this projection property as well and indeed, this turns out to be the case. The bootstrap principle, whereby the S -matrix for a bound state is obtained from the S -matrices of the constituent particles,

(4.2)

is a consequence of the property (1.13) of the universal R-matrix with respect to the coproduct.

There is a famous no-go theorem (S. Coleman and J. Mandula, Phys.Rev. 159 (1967) 1251) which states the “impossibility of combining space-time and internal symmetries in any but a trivial way”. Affine quantum group symmetry circumvents this no-go theorem. In fact, the derivation D is the infinitesimal two-dimensional Lorentz boost generator and the other symmetry charges transform non-trivially under these Lorentz transformations, see (1.1).

The non-cocommutative coproduct (1.7) means that a $U_h(\hat{\mathfrak{g}})$ symmetry generator, when acting on a two-soliton state, acts differently on the left soliton than on the right soliton. This is only possible because the generator is a non-local symmetry charge – that is, a charge which is obtained as the space integral of the time component of a current which itself is a non-local expression in terms of the fields of the theory.

Similarly, many nonlinear sigma models possess non-local charges which form $Y(\mathfrak{g})$, and the construction proceeds similarly, now utilising rational R -matrices, and with particle multiplets forming fundamental

representations of $Y(\mathfrak{g})$. In each case the three-point couplings corresponding to the formation of bound states, and thus the analogues for $U_h(\hat{\mathfrak{g}})$ and $Y(\mathfrak{g})$ of the Clebsch-Gordan couplings, obey a rather beautiful geometric rule originally deduced in simpler, purely elastic scattering models [2].

More details about this topic can be found in [4, 12].

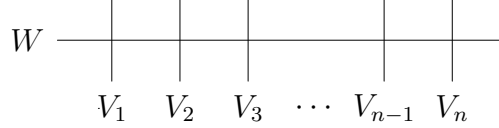
5. INTEGRABLE QUANTUM SPIN CHAINS

Affine quantum groups provide an unlimited supply of integrable quantum spin chains. From any R-matrix $R(\theta)$ for any tensor product of finite-dimensional representations $W \otimes V$ one can produce an integrable quantum system on the Hilbert space $V^{\otimes n}$. This Hilbert space can then be interpreted as the space of n interacting spins. The space W is an auxiliary space required in the construction but not playing a role in the physics.

Given an arbitrary R-matrix $R(\theta)$ one defines the monodromy matrix $T(\theta) \in \text{End}(W \otimes V^{\otimes n})$ by

$$T(\theta) = R_{01}(\theta - \theta_1) R_{02}(\theta - \theta_2) \cdots R_{0n}(\theta - \theta_n)$$

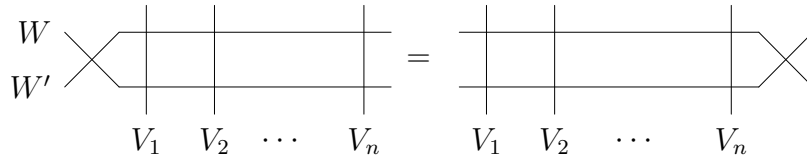
where, as usual, R_{ij} is the R-matrix acting on the i -th and j -th component of the tensor product space. The θ_i can be chosen arbitrarily for convenience. Graphically the monodromy matrix can be represented as



As a consequence of the Yang-Baxter equation satisfied by the R-matrices the monodromy matrix satisfies

$$(5.1) \quad RTT = TTR.$$

or, graphically,



One defines the transfer matrix

$$\tau(\theta) = \text{tr}_W T(\theta)$$

which is now an operator on $V^{\otimes n}$, the Hilbert space of the quantum spin chain. Due to (5.1) two transfer matrices commute,

$$[\tau(\theta), \tau(\theta')] = 0$$

and thus the $\tau(\theta)$ can be seen as a generating function of an infinite number of commuting charges, one of which will be chosen as the Hamiltonian. This Hamiltonian can then be diagonalized using the algebraic Bethe Ansatz.

One is usually interested in the thermodynamic limit where the number of spins goes to infinity. In this limit, it has been conjectured, the Hilbert space of the spin chain carries a certain infinite-dimensional representation of the quantum affine algebra and this has been used to solve the model algebraically, using vertex operators [10].

6. BOUNDARY QUANTUM GROUPS

In applications to physical systems that have a boundary the Yang-Baxter equation (0.1) appears in conjunction with the boundary Yang-Baxter equation, also known as the reflection equation,

(6.1)

$$R_{12}(u-v)K_1(u)R_{21}(u+v)K_2(v) = K_2(v)R_{12}(u+v)K_1(u)R_{21}(u-v).$$

The matrices K are known as reflection matrices. This equation was originally introduced by Cherednik to describe the reflection of particles of a boundary in an integrable scattering theory and was used by Sklyanin to construct integrable spin chains and quantum field theories with boundaries.

Boundary quantum groups are certain co-ideal subalgebras of affine quantum groups. They provide the algebraic structures underlying the solutions of the boundary Yang-Baxter equation in the same way in which affine quantum groups underlie the solutions of the ordinary Yang-Baxter equation. Both allow one to find solutions of the respective Yang-Baxter equation by solving a linear intertwining relation. In the case without spectral parameters these algebras appear in the theory of braided groups [See articles 28, 46].

For example the subalgebra $B_\epsilon(\hat{\mathfrak{g}})$ of $U_h(\hat{\mathfrak{g}}')$ generated by

$$(6.2) \quad Q_i = q_i^{H_i/2}(E_i^+ + E_i^-) + \epsilon_i(q_i^{H_i} - 1), \quad i = 0, \dots, r$$

is a boundary quantum group for certain choices of the parameters $\epsilon_i \in \mathbb{C}[[h]]$. It is a left coideal subalgebras of $U_h(\hat{\mathfrak{g}}')$ because

$$(6.3) \quad \Delta(Q_i) = Q_i \otimes 1 + q_i^{H_i} \otimes Q_i \in U_h(\hat{\mathfrak{g}}') \otimes B_\epsilon(\hat{\mathfrak{g}}).$$

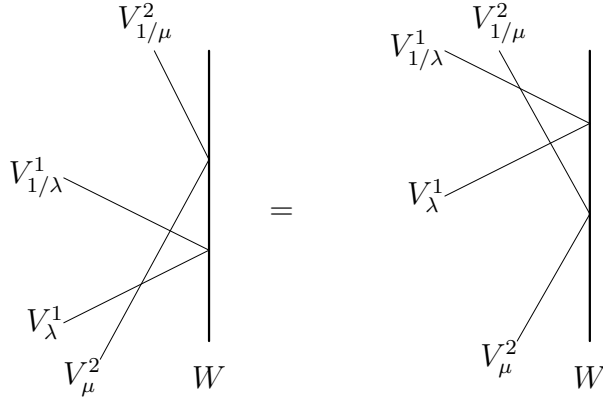
Intertwiners $K(\lambda) : V_{\eta\lambda} \rightarrow V_{\eta/\lambda}$ for some constant η satisfying

$$(6.4) \quad K(\lambda)\rho_{\eta\lambda}(Q) = \rho_{\eta/\lambda}(Q)K(\lambda), \quad \text{for all } Q \in B_\epsilon(\hat{\mathfrak{g}}),$$

provide solutions of the reflection equation in the form

$$(6.5) \quad \begin{aligned} & (\text{id} \otimes K^2(\mu))\check{R}^{12}(\lambda\mu)(\text{id} \otimes K^1(\lambda))\check{R}^{21}(\lambda/\mu) \\ &= \check{R}^{12}(\lambda/\mu)(\text{id} \otimes K^1(\lambda))\check{R}^{21}(\lambda\mu)(\text{id} \otimes K^2(\mu)). \end{aligned}$$

This can be expanded to the case where the boundary itself carries a representation W of $B_\epsilon(\hat{\mathfrak{g}})$. The boundary Yang-Baxter equation can be represented graphically as



Another example is provided by twisted Yangians where, when the I_a and J_a are constructed as non-local charges in sigma models, it is found that a boundary condition which preserves integrability leaves only the subset

$$I_i \quad \text{and} \quad \tilde{J}_p = J_p + \frac{1}{4}f_{piq}(I_i I_q + I_q I_i)$$

conserved, where i labels the \mathfrak{h} -indices and p, q the \mathfrak{k} -indices of a symmetric splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$. The algebra $Y(\mathfrak{g}, \mathfrak{h})$ generated by the I_i, \tilde{J}_p is, like $B_\epsilon(\hat{\mathfrak{g}})$, a coideal subalgebra, $\Delta(Y(\mathfrak{g}, \mathfrak{h})) \subset Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h})$, and again yields an intertwining relation for K -matrices. For $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{h} = \mathfrak{so}_n$ or \mathfrak{sp}_{2n} , $Y(\mathfrak{g}, \mathfrak{h})$ is the *twisted Yangian* described in [13].

All the constructions in earlier sections of this review have analogs in the boundary setting. For more details see [3, 12].

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